Berlin Mathematics Research Center MATH+

Tropical Mechanism Design

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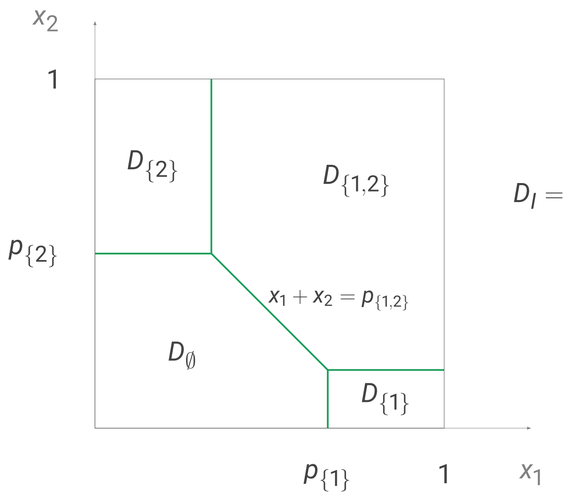
Intoduction

- We want to sell n objects to a single buyer
- The buyer has a valuation vector $x \in [0,1]^n$, drawn according to a distribution f
- A price schedule is a collection $\{p_I\}_{I\subset[n]}$ of prices, one for each bundle of items $I \subset [n]$
- The buyer takes the bundle maximizing her utility $u(x) = \max\left\{\sum_{i\in I} x_i - p_I \mid I \subset [n]\right\}$
- The utility is a max-tropical polynomial cutting the valuation space $[0,1]^n$ into regions D_I for each bundle I



$$D_I = \left\{ x \in [0,1]^n \;\middle|\; I \text{ maximizes } u(x) \right\}$$

Goal: Find a price schedule $p = \{p_I\}_{I\subset[n]}$ maximizing the seller's revenue

$$\mathrm{Rev}(p) = \sum_{I\subset[n]} \int_{x\in D_I} p_I \, df \quad .$$

Optimal Auctions

- Consider the uniform case $f = \mathcal{U}_{[0,1]^n}$, then we get $\mathrm{Rev}(p) = \sum_{I\subset[n]} p_I \operatorname{vol}_n D_I$
- We demand p to be submodular, i.e., $p_{I\cup J} + p_{I\cap J} \leq p_I + p_J$, else a buyer interested in buying a bundle $I \cup J$ would just buy the bundles I and J separately and sell off the bundle $I \cap J$

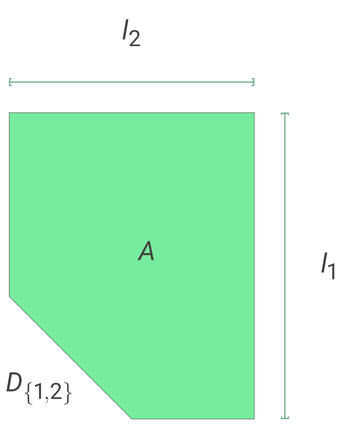
Lemma: If the price schedule p is submodular, then there is a $\beta_I > 0$ s.t.

$$\beta_I \frac{\partial \mathrm{Rev}(p)}{\partial p_I} = \operatorname{vol}_{|I|} \bar{D}_I - \frac{1}{n+1} \sum_{i\in I} \operatorname{vol}_{|I|-1} \bar{D}_I^i =: \delta_n(I),$$

where $\bar{D}_I$ is D_I projected onto $\mathbb{R}^I$ and $\bar{D}_I^i = \bar{D}_I \cap \{x \mid x_i = 1\}$.

$$\frac{\partial \mathrm{Rev}(p)}{\partial p_{\{1,2\}}} = \delta_2(\{1,2\}) = A - \frac{2}{3}(l_1 + l_2)$$

(In this case, $\beta_{\{1,2\}} = 1$)



Necessary Conditions for Optimality

The revenue function can obtain it's maximum only if the critical point conditions are satisfied. Using the last lemma they read:

$$\text{If } \operatorname{vol}_n D_I > 0, \quad \text{then } \delta_n(I) = 0$$

Proposition: Assume $p_{\{i\}} < 1$ for all $i \in [n]$ and $\delta_n(J) = 0$ for all $J \subsetneq I$, then

$$\delta_n(I) = 0 \;\Leftrightarrow\; \operatorname{vol}_{|I|} \pi_I(D_\emptyset) = 1 - \tfrac{|I|}{n+1}$$

where π_I is the projection onto $\mathbb{R}^I$.

Straight Jacket Auction

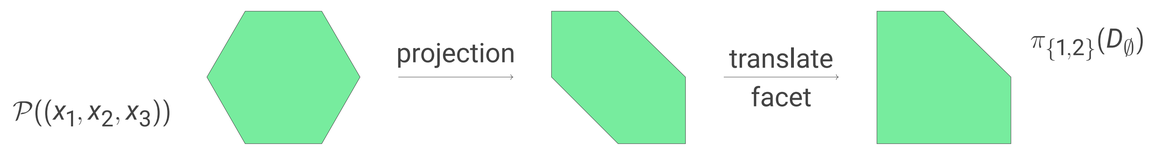
- The Straight Jacket Auction (SJA) is symmetric, i.e., bundles with the same cardinality k have the same price p_k
- Compute $p_1, \ldots, p_n$ sequentially such that $\operatorname{vol}_k \pi_{[k]}(D_\emptyset) = 1 - \frac{k}{n+1}$ is satisfied for $p_1, \ldots, p_k$, we call such a partial price schedule $(p_1, \ldots, p_k)$ *feasible*
- There is always some k s.t. there are feasible prices $p_1, \ldots, p_k$ and a price p_n with $\operatorname{vol}_n D_\emptyset = \frac{1}{n+1}$
- The SJA is defined by the prices where this k is maximized. The prices $p_{k+1}, \ldots, p_{n-1}$ are set to p_n, the corresponding regions vanish

Theorem (Joswig, Klimm, Spitz [2])

The SJA is the only deterministic, symmetric and submodular auction satisfying the necessary conditions, except for eventually some degenerate cases.

Generalized Permutahedra

- The permutahedron $\mathcal{P}(x)$ is the convex hull of all permutations of x
- Translating its facets results in a generalized permutahedron
- For all I, $\pi_I(D_\emptyset)$ is a (projection of a) generalized permutahedron



- For symmetric prices $p = (p_1, \ldots, p_n)$, the projections of $D_\emptyset$ are called SIM-bodies and already encode the prices

$$\pi_{[k]}(D_\emptyset) = \left\{ x \in \mathbb{R}^k_{\geq 0} \;\middle|\; \sum_{i\in I} x_i \leq p_{|I|} \text{ for } I \subset [k] \right\}$$

Volume Polynomials

Being a generalized permutahedron, the volume of the SIM-body $\operatorname{vol}_{|I|} \pi_I(D_\emptyset)$ is a homogeneous polynomial in $p_1, \ldots, p_n$, which can be computed in several ways:

a) Recursive integration method as used by Giannakopoulos and Koutsoupias [1]
b) Adaptation of Lawrence's formula [3]
c) Counting bipartite matching method derived from Postnikov [4]

Computations

The SJA-prices $p = (p_1, \ldots, p_n)$ can be computed as the solution to the polynomial system

$$1 - \frac{k}{n+1} \;=\; \operatorname{vol}_k \pi_{[k]}(D_\emptyset) \;\text{ for }\; k \in [n] \quad ,$$

which satisfy $p_1 \leq 1$ and the submodularity inequalities

$$p_k - p_{k-1} \;\leq\; p_{k-1} - p_{k-2} \;\text{ for }\; k \in \{2, \ldots, n\} \quad .$$

It may be necessary to modify the polynomial system, when the submodularity conditions cannot be satisfied otherwise.

Example for $n = 3$:

$$\begin{aligned} 1 - \frac{1}{4} &= p_1 \\ 1 - \frac{2}{4} &= -p_1^2 + 2p_1p_2 - \frac{1}{2}p_2^2 \\ 1 - \frac{3}{4} &= \frac{1}{2}p_1^3 - \frac{3}{2}p_1^2p_3 - 3p_1p_2^2 + 6p_1p_2p_3 - \frac{3}{2}p_1p_3^2 + p_2^3 - \frac{3}{2}p_2^2p_3 + \frac{1}{6}p_3^3 \end{aligned}$$

In the symmetric case, the revenue can be computed via:

$$\mathrm{Rev}(p) = \sum_{i\in[n]} \binom{n}{i} p_i \operatorname{vol}_n D_{[i]}$$

For the SJA, $D_{[i]}$ is congruent to the product of SIM-bodies, hence we can use the same volume polynomials as for the prices.

Implementing the recursive integration method to compute the polynomials for the volume of the SIM-bodies, we were able to compute and verify the prices and the revenue for up to 12 items (known before for up to 6)

n	$p_1^{(n)}$	$p_2^{(n)}$	$p_3^{(n)}$	$p_4^{(n)}$	$p_5^{(n)}$	$p_6^{(n)}$	$p_7^{(n)}$	$p_8^{(n)}$	$p_9^{(n)}$	$p_{10}^{(n)}$	$p_{11}^{(n)}$	$p_{12}^{(n)}$
1	1/2											
2	2/3	0.862										
3	3/4	1.146	1.226									
4	4/5	1.317	1.581	1.601								
5	5/6	1.431	1.817	–	1.986							
6	6/7	1.512	1.986	2.286	–	2.377						
7	7/8	1.573	2.113	2.500	2.739	–	2.775					
8	8/9	1.621	2.211	2.667	2.991	–	–	3.178				
9	9/10	1.659	2.290	2.800	3.192	3.466	–	–	3.584			
10	10/11	1.690	2.355	2.909	3.356	3.696	3.932	–	–	3.995		
11	11/12	1.715	2.409	3.000	3.493	3.888	4.188	4.392	–	–	4.408	
12	12/13	1.737	2.454	3.077	3.609	4.051	4.404	4.670	–	–	–	4.825

"–" means that the corresponding bundle is not sold

n	1	2	3	4	5	6	7	8	9	10	11	12
Rev	0.25	0.549	0.875	1.220	1.576	1.943	2.318	2.699	3.086	3.478	3.883	4.336

References

[1] Y. Giannakopoulos and E. Koutsoupias.
Duality and optimality of auctions for uniform distributions.
SIAM J. Comput., 47(1):121–165, 2018.
URL: https://doi.org/10.1137/16M1072218, doi:10.1137/16M1072218.

[2] M. Joswig, M. Klimm, and S. Spitz.
Generalized permutahedra and optimal auctions, 2021.
arXiv:2108.00979.

[3] J. Lawrence.
Polytope volume computation.
Math. Comp., 57(195):259–271, 1991.
URL: https://doi.org/10.2307/2938672, doi:10.2307/2938672.

[4] A. Postnikov.
Permutohedra, associahedra, and beyond.
Int. Math. Res. Not. IMRN, (6):1026–1106, 2009.
URL: https://doi.org/10.1093/imrn/rnn153, doi:10.1093/imrn/rnn153.